

Math 243
Summer 2018
Practice Exam 1
Doomsday
Time Limit: Probably Not Enough

Name (Print): Key

Problem	Points	Score
1	15	
2	20	
3	30	
4	15	
5	15	
6	20	
7	20	
Total:	135	

1. (15 points) Let $P = (1, 2, 3)$ and $Q = (0, -1, 2)$.

a) Find the distance between P and Q .

$$\begin{aligned} D &= \sqrt{(1-0)^2 + (2-(-1))^2 + (3-2)^2} \\ &= \sqrt{11} \end{aligned}$$

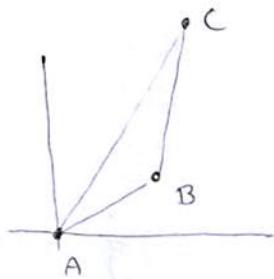
b) Give the equation of a sphere, centered at P , that has the point Q on its surface.

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 11$$

c) Find the vector \overrightarrow{PQ} .

$$\begin{aligned} \overrightarrow{PQ} &= \vec{Q} - \vec{P} \\ &= \langle -1, -3, -1 \rangle \end{aligned}$$

2. (a) (10 points) The following points define the vertices of a triangle $A = (0, 0)$, $B = (1, 3)$ and $C = (2, 7)$. Find the measure of angle B .

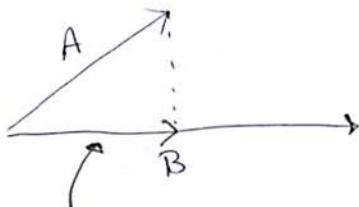


$$\overrightarrow{BA} = \langle -1, -3 \rangle, \quad |\overrightarrow{BA}| = \sqrt{10}$$

$$\overrightarrow{BC} = \langle 1, 4 \rangle, \quad |\overrightarrow{BC}| = \sqrt{17}$$

$$\begin{aligned} m\angle B &= \cos^{-1} \left(\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} \right) \\ &= \cos^{-1} \left(\frac{-1 \cdot 1 + (-3) \cdot 4}{\sqrt{170}} \right) = \cos^{-1} \left(\frac{-13}{\sqrt{170}} \right) \end{aligned}$$

- (b) (10 points) Let $u = i + j$, $v = i + j + k$. Find the projection of u onto $u + v$.



$$\begin{aligned} \text{This is } \text{Proj}_B A &= \cos(\theta) |A| \frac{B}{|B|} \\ &= \frac{\cos(\theta) |A| |B|}{|B|^2} B \\ &= \frac{A \cdot B}{|B|^2} B \end{aligned}$$

$$u+v = 2i + 2j + k, \quad |u+v| = \sqrt{4+4+1} = 3$$

$$\text{Proj}_{u+v} u = \frac{u \cdot (u+v)}{|u+v|^2} \cdot (u+v) = \frac{4}{9} (2i + 2j + k)$$

3. (a) (10 points) The vectors $u = 1i + 2j$ and $v = j + 3k$ lie in a plane. Give an equation of the plane that goes through the point $P = (1, 0, 1)$ and is parallel to the aforementioned plane.

Normal Vector is $u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix}$

$$= 6i - (3)j + (+k)$$

$$= 6i - 3j + k$$

Equation: $6(x-1) + (-3)(y-0) + (1)(z-1) = 0$

- (b) (10 points) Give the equation of a line, perpendicular to the plane $2x + 3y + z = 6$, that goes through the point $(1, 0, 1)$.

$$\mathbf{N} = \langle 2, 3, 1 \rangle$$

$$x = 1 + 2t$$

$$y = 0 + 3t$$

$$z = 1 + t$$

- (c) (10 points) In part b) find the point in space the line hits the plane.

$$2(1+2t) + 3(0+3t) + (1+t) = 6$$

$$\Rightarrow 2 + 4t + 9t + 1 + t = 6$$

$$\Rightarrow 14t = 3$$

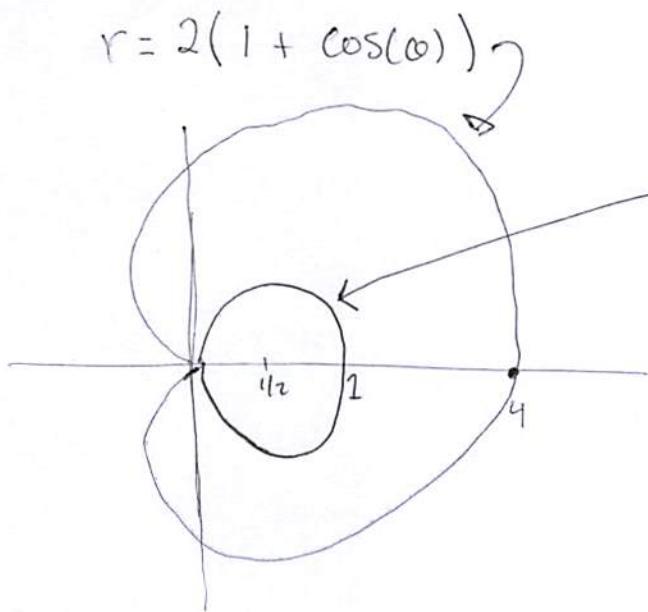
$$t = \frac{3}{14}$$

So, $x = 1 + 2(\frac{3}{14})$

$y = 3(\frac{3}{14})$ is on the plane, and
 $z = 1 + \frac{3}{14}$ on the line.

4. (15 points) This page will be similar to problems 1-12 in 11.6.

5. (15 points) Consider the polar coordinate equations $r = 2(1 + \cos(\theta))$ and $r = \cos(\theta)$. Graph both of these, and find their arclength. Find the area between the two graphs. Note that this is non-standard...



$$r = \cos(\theta)$$

Arc Length of $r = \cos(\theta)$

$$\int_0^\pi \sqrt{\cos^2(\theta) + \sin^2(\theta)} d\theta \\ = \pi$$

$$\text{Area of circle: } \pi (\frac{1}{2})^2 \\ = \frac{\pi}{4}$$

Arc length of $r = 2(1 + \cos(\theta))$

$$\begin{aligned} A.L. &= \int_0^{2\pi} \sqrt{4(1+\cos(\theta))^2 + 4\sin^2(\theta)} d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 4 + 8\cos(\theta)} d\theta \\ &= \sqrt{8} \int_0^{2\pi} \sqrt{1 + \cos(\theta)} d\theta \\ &= \sqrt{8} \int_0^{2\pi} \sqrt{2\cos^2(\frac{\theta}{2})} d\theta \\ &= \sqrt{8} \int_0^{2\pi} \sqrt{2} |\cos(\theta/2)| d\theta \\ &= 8 \left(\int_0^\pi \cos(\theta/2) d\theta - \int_\pi^{2\pi} \cos(\theta/2) d\theta \right) \\ &= 16 \end{aligned}$$

So, the area between the curves is

$$6\pi - \frac{\pi}{4}$$

~~2~~ $2\cos^2(\theta) = 1 + \cos(2\theta)$

Area of $r = 2(1 + \cos(\theta))$:

$$A = \int_0^{2\pi} \frac{1}{2} (2(1 + \cos(\theta)))^2 d\theta = 2 \int_0^{2\pi} 1 + 2\cos(\theta) + \cos^2(\theta) d\theta = 2(2\pi + 0 + \pi) \\ = 6\pi$$

6. (a) (10 points) Let $r(t) = t^2 \mathbf{i} + (1 - 2t^2) \mathbf{j} + 10 \mathbf{k}$. Give the equation of the tangent line when $t = 2$.

$$\mathbf{r}'(t) = 2t \mathbf{i} + (-4)t \mathbf{j} + 0\mathbf{k}$$

$$\mathbf{r}(2) = 4\mathbf{i} + (-7)\mathbf{j} + 10\mathbf{k}, \quad \mathbf{r}'(2) = 4\mathbf{i} + (-8)\mathbf{j}$$

Tangent Line: $x = 4 + 4t$

$$y = -7 + (-8)t$$

$$z = 10$$

- (b) (10 points) Find the arclength of the above from $t = 0$ to $t = 1$.

$$\begin{aligned} \text{A.J.} &= \int_0^1 \sqrt{(2t)^2 + (-4t)^2 + 0^2} \, dt \\ &= \int_0^1 \sqrt{4t^2 + 16t^2} \, dt \\ &= \sqrt{20} \int_0^1 t \, dt \\ &= \sqrt{20} \left[\frac{t^2}{2} \right]_0^1 \end{aligned}$$

7. (20 points) Let $r(t) = t \cos(t) \mathbf{i} + t \sin(t) \mathbf{j} + t \mathbf{k}$. Describe this path in your own, freaky words. Find the unit normal vector, T , the curvature, κ , and the principle unit normal N . For extra credit, determine the unit binormal vector B .

Path: It's a tornado. important

$$\mathbf{v}(t) = \cos(t) \mathbf{i} + t(-\sin(t)) \mathbf{i} + \sin(t) \mathbf{j} + t \cos(t) \mathbf{j} + 1 \mathbf{k}$$

$$|\mathbf{v}(t)| = \sqrt{(\cos(t) + t(-\sin(t)))^2 + (\sin(t) + t \cos(t))^2 + 1^2}$$

$$= \sqrt{1 + t^2 + 1 + \underbrace{\text{stuff}}_1}$$

So, $T = \frac{\cos(t) - t \sin(t)}{\sqrt{2+t^2}} \mathbf{i} + \frac{\sin(t) + t \cos(t)}{\sqrt{2+t^2}} \mathbf{j} + \frac{1}{\sqrt{2+t^2}} \mathbf{k}$

$$\begin{aligned} \frac{dT}{dt} &= \left[\frac{[-\sin(t) - (\sin(t) + t \cos(t))] \sqrt{2+t^2} - \frac{1}{2\sqrt{2+t^2}} (2t)(\cos(t) - t \sin(t))}{2+t^2} \right] \mathbf{i} \\ &+ \left[\frac{(\cos(t) + \cos(t) + t(-\sin(t))) \sqrt{2+t^2} - \frac{1}{2\sqrt{2+t^2}} (2t)(\sin(t) + t \cos(t))}{2+t^2} \right] \mathbf{j} \\ &+ -\frac{1}{2} (2+t^2)^{-\frac{3}{2}} \cdot 2t \mathbf{k} \end{aligned}$$

So, clearly, $\kappa = \frac{1}{|\mathbf{v}(t)|} \cdot \left| \frac{dT}{dt} \right|$

$$\text{and } N = \frac{\frac{dT}{dt}}{\left| \frac{dT}{dt} \right|}$$

(Sorry this problem is so stupid)